3 Guided waves in optical waveguides

Ridge (rib) Waveguide

Multi-Mode Graded-Index Fiber  Single-Mode Step-Index Fiber
Goals of the chapter:

- Description of light propagation in dielectric waveguide structures
- Relation of the guided wave properties $\vec{E}(\vec{r},t); \vec{H}(\vec{r},t); \beta(\omega)$ to the geometric and dielectric structure of the waveguide
- Typical properties of dielectric waveguide structures for optical communication
- Dispersion effects and pulse broadening due to the waveguide structure (in contrast to dispersion due to the frequency response of the dielectric material)

Methods for the Solution:

- Propagation of “classical” light can be described as a electromagnetic (EM) wave obeying the Maxwell field and material equations
- Solution of Maxwell’s equation with the boundary conditions given be the structure of the waveguide
- Determination of the solutions (modes) of the time- and space dependent EM vector fields $\vec{E}(\vec{r},t), \vec{B}(\vec{r},t), \vec{H}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$ and the propagation constant $\vec{k}(\omega)$ resp. $\beta(\omega)$ as a function of the optical frequency $\omega$
- Find modal dispersion $D_{\text{mode}}(\omega)$ from $\beta(\omega)$ for harmonic waves

Remark:
The basic material on Maxwell’s equations and dielectric waveguides has been treated in the courses “Fields and Components I, II” by Prof. R.Vahldieck/Dr. P.Leuchtmann in the 3. and 4.semester. We will only make references where the topic can be found in the scripts.
3 Guides Waves in optical Waveguides

3.1 Guiding Lightwaves – historical Overview

1) Lens waveguides: (Gobau 1960)
Light beams can be formed and propagated by lens and mirror systems

but, light beam in free space are broadened by **diffraction** (beam widening) and need to be refocused by lenses. Diffraction effects in light beams increase with decreasing beam diameter \( A \).

2) metallic waveguides:
   Possible but losses at optical frequencies are too high, technologic challenge for waveguide dimensions on the \( \mu \text{m-scale} \)

3) dielectric waveguides: (1966 – today)
   Dielectric materials like glasses and liquids can have **low absorption and scattering losses** at optical frequencies
   **Fabrication technology** for wave guide dimension comparable to the optical wavelength \( \lambda \sim 1\mu \text{m} \) feasible
Conceptual idea of light guiding in dielectric structures: (ray optic picture)

use lossless **total reflections** at interfaces of 2 dielectrics (solids or liquids) with refractive indices $n_2$ and $n_1$ where $n_1 > n_2$

a) planar WG (2D-propagation)

b) cylindrical fiber WG (1D-propagation)

- Liquid Core Glass fiber (Uni South Hampton, before 1970)
- Glass fibers (after 1970)
Glass fiber drawing process: draw large preforms into small fibers

1) Preform fabrication:
   - Preform fabrication: draw large preforms into small fibers
   - Requirements for optical glass fibers:
     - Ultra pure materials for low light absorption below –1dB/km glass
     - Precise geometry control low 0.1µm of ~5-10µm core diameter
     - Homogenous and precise material composition
     - High interface quality, low interface roughness, low scattering
     - Fabrication of single fiber length of several km
Loss mechanisms in silica optical glass fibers:

- **Absorption by impurities**, mainly OH-radicals at 0.95, 1.23 and 1.39 μm wavelength
- Sub-wavelength **density fluctuations** ($\Delta l < \lambda$) → Raleigh-Scattering $\sim 1/\lambda$.
- **UV-Absorption** by electron excitation in the SiO$_2$-complex at $\sim 0.3$ μm
- **IR-absorption** by Si-O-vibrations at $\sim 5$ μm
- Geometrical **form fluctuations** ($\Delta l > \lambda$), Mie-Scattering, microbending

Low water content fibers have a continuous band from 1200-1700 nm.
3.2 Ray Optics (geometrische Strahlenoptik)

Light propagation in fibers with core diameters $d$ much larger than the optical wavelength $\lambda$ ($d \gg \lambda$) can be approximated by the propagation and diffraction of light beams (rays).

In the limit of ray optics light propagation in fibers can be modeled by the Total Reflection at the Interface between fiber core ($n_1$) and the fiber cladding ($n_2$): lossless Zig-Zag-Transmission

Snells Law of refraction and reflection:

$$\frac{\cos(\varphi_2)}{\cos(\varphi_1)} = \frac{n_1}{n_2} > 0 \quad \text{and} \quad \frac{n_1}{n_2} < 1$$

**Critical Angle $\varphi_c$ for total reflection** ($\varphi_2=0$, $\cos\varphi_2 = 1$) at a dielectric interface with refractive indices $n_1 > n_2$:

$$\cos(\varphi_c) = \frac{n_2}{n_1} < 1 \quad ; \quad \varphi_1 < \varphi_c \rightarrow \text{total reflection} \quad (\text{no transmission})$$

$$\varphi_1 > \varphi_c \rightarrow \text{total reflection and transmission}$$

**Acceptance Angle $\varphi_a$:**

At the entrance interface ($n_o, n_1$) of the fiber the incoming beams ($\varphi_0$) are refracted according to Snells-Law:

$$n_o \sin(\varphi_a) = n_1 \sin(\varphi_c) = n_1 \sqrt{1 - \left(\frac{n_2}{n_1}\right)^2} = \sqrt{n_1^2 - n_2^2} \quad (\text{limiting situation for total reflection } \varphi_0 = \varphi_a)$$
All beams with an entrance angle $\varphi_0 < \varphi_a$ are propagated lossless by total reflections through a straight fiber. Beams with $\varphi_0 > \varphi_a$ suffer refraction losses into the cladding and are damped.

For fiber characterization the **numerical aperture NA** is defined as a figure of merit:

$$\text{NA} = \sin(\varphi_a) = \frac{1}{n_0} \sqrt{n_1^2 - n_2^2} \quad \text{with} \quad \Delta = \frac{n_1 - n_2}{n_1}$$

$\Delta$ relative refractive index difference between core and cladding

**Conclusions from the simple ray-model of the optical fiber:**

- All light beams entering the straight fiber within the cone defined by the NA are transmitted lossless by total reflections and exit the fiber within the NA-cone.

- Large relative index differences $\Delta$ between fiber core and cladding result in large NAs with high coupling efficiency between a light source and a fiber, but the time delays $\Delta t$ between the different Zig-Zag-beams becomes large (fiber signal dispersion) trade-off $\Delta, d \leftrightarrow \Delta t$

  Multimode step index fiber: $\Delta = 1 - 3\% \quad \text{NA} \sim 0.4$

  Single mode step index fibers $\Delta = 0.2 - 1\% \quad \text{NA} = 0.1 - 0.2$, $\varphi_a = 12.2^\circ$ mit $n_1 = 1.5$ and $\Delta = 1\%$

- Strong bending of the fiber can result in a violation of the total reflection condition at the bends and the light beams can exit the fiber core (bending losses).

The ray model fails if $\lambda \sim d$ (low order modes) and does not provide the light intensity distributions (mode intensity profiles)

**needed: wave description of light propagation**
3.3 Wave propagation in cylindrical optical waveguides

Waveguides for signal transmission must propagate a wave longitudinally in the z-direction ($\beta_z(\omega)$) and should confine (resonance-like) in the wave in the transverse T-direction (x,y-plane).

How will the transverse confinement of the wave influence the propagation in the z-direction?

Dielectric optical waveguides have a cylindrical structure, with a homogenous refractive index n in the z direction ($n(x,y) \neq n(z)$). The refractive index in the transverse direction is inhomogeneous, $n(x,y)$ for transverse confinement.

Goal:

find the possible EM-waves $\vec{E}(\vec{r},t), \vec{H}(\vec{r},t)$ supported by the cylindrical WG-structure with a transverse index profile $n(x,y)$ at a frequency $\omega$, which are solutions of the Maxwell’s Equations.

Here we consider the EM-field as a classical unquantized wave, because we are interested in propagation properties. However photons, which are the energy quantum $E = h\omega$ of the optical field, show a dual behaviour depending on the experiment:

- particle-like for interactions, resp. energy exchange with matter
- wave-like for propagation
3.3.1 Maxwell’s-Equations for $\vec{E}(\vec{r}, t), \vec{H}(\vec{r}, t)$-waves in cylindrical dielectric WGs:

Assumptions:

- dielectric is free of fixed space charges, $\rho = 0$
- there are no convection currents flowing in the dielectric (isolator), $\vec{j} = \sigma \vec{E} = 0$
- the dielectrics are isotropic, $\varepsilon = \varepsilon_0 \varepsilon_r$ ; $\varepsilon_r = \text{scalar}$
- the dielectrics are non-magnetic, $\mu = \mu_0$ ; $\mu_r = 1$

1) Separation of the longitudinal (z) and lateral (x,y) geometry:

$\vec{r} = (x, y, z) = \vec{r}_r + \vec{r}_z = \vec{r}_r + r_r \vec{e}_z$

$\vec{r}_r$ and $\vec{r}_z$ are orthogonal : $\vec{r}_r \cdot \vec{r}_z = 0$

2) Maxwell Vector-Field Equations (MW):

\[
\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} ; \quad \nabla \times \vec{H} = -\varepsilon \frac{\partial \vec{E}}{\partial t}
\]

$\nabla \vec{E} = 0$ ; $\nabla \vec{H} = 0$

with $\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$

3) Materials Equations:

$\vec{D} = \varepsilon \vec{E}$

$\vec{B} = \mu_0 \vec{H}$

$\Rightarrow$ 6 field variables: $E_x(\vec{r}, t), E_y(\vec{r}, t), E_z(\vec{r}, t)$ and $H_x(\vec{r}, t), H_y(\vec{r}, t), E_x(\vec{r}, t), H_z(\vec{r}, t)$,
Elimination of one field variable from MW’s equations leads to the

**Homogeneous Vector Wave Equations:**  (derivation see F&K I)

\[
\begin{align*}
\left( \Delta - \mu \varepsilon \frac{\partial^2}{\partial t^2} \right) \vec{E}(\vec{r}, t) &= 0 ; \\
\left( \Delta - \mu \varepsilon \frac{\partial^2}{\partial t^2} \right) \vec{H}(\vec{r}, t) &= 0 \quad \text{with} \quad \Delta = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)
\end{align*}
\]

Assuming that the fields are excited by sources with a **harmonic time dependence** \( e^{+i\omega t} \):

**4) Harmonic field solutions with a separation of space \( \vec{r} \) and time \( t \) dependence:**

\[
\vec{E}(\vec{r}, t) = Re \{ \vec{E}(\vec{r}) e^{i\omega t} \} = \frac{1}{2} \vec{E}(\vec{r}) e^{i\omega t} + \frac{1}{2} \vec{E}^*(\vec{r}) e^{-i\omega t}
\]

\[
\vec{H}(\vec{r}, t) = Re \{ \vec{H}(\vec{r}) e^{i\omega t} \} = \frac{1}{2} \vec{H}(\vec{r}) e^{i\omega t} + \frac{1}{2} \vec{H}^*(\vec{r}) e^{-i\omega t}
\]

**5) Homogeneous Helmholtz Equation for the spatial Functions \( \vec{E}(\vec{r}) ; \vec{H}(\vec{r}) \):**

For harmonic time dependence the time-derivation operators transform as

\[
\begin{align*}
\frac{\partial}{\partial t} &\quad \longrightarrow \quad + i\omega \\
\frac{\partial^2}{\partial t^2} &\quad \longrightarrow \quad - \omega^2
\end{align*}
\]

**Spatial \( z,T \)-separation (transversal, \( x,y \) of operators and vectors:** (in cartesian coordinates)

**Definition:**

\[
\Delta = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \left( \frac{\partial^2}{\partial z^2} \right) = \Delta_T + \Delta_z
\]
\[
(\Delta_T + \Delta_z + k^2)\vec{E} = 0 ; \quad (\Delta_T + \Delta_z + k^2)\vec{H} = 0
\]

formal decomposition:  \( k^2 = k_T^2 + k_Z^2 = \mu \varepsilon \omega^2 \)  \( (k_T, k_Z \text{ is not defined yet}) \)

\[
(\Delta_T + \Delta_z + k_T^2 + k_Z^2)\vec{E}(\vec{r}) = 0 \quad ; \quad (\Delta_T + \Delta_z + k_T^2 + k_Z^2)\vec{H}(\vec{r}) = 0
\]

**Solution-“Ansatz” for the Helmholtz-Equation for a \( z \)-guided wave:**

If we decompose the spatial propagation vector \( \vec{k} \) of the solution in \( z \)- and \( T \)-components:

with \( \vec{k} = \vec{k}_T + \vec{k}_Z = \vec{k}_T + k_Z \vec{e}_Z \quad \rightarrow \quad |\vec{k}|^2 = |\vec{k}_T|^2 + |\vec{k}_Z|^2 = k^2 \quad \text{and} \quad |\vec{k}_T|^2 = k_T^2, |\vec{k}_Z|^2 = k_Z^2 \)

then a \( z \)- and \( T \)-decomposition of the field vectors is plausible:

**Transversal and longitudinal field decomposition for a plan wave:**

\[
\vec{E}(\vec{r}) = \left( E_x(\vec{r}), E_y(\vec{r}), E_z(\vec{r}) \right) = \left( E_x(\vec{r}), E_y(\vec{r}), 0 \right) + \left( 0, 0, E_z(\vec{r}) \right) = \vec{E}_T(\vec{r}) + \vec{E}_Z(\vec{r})
\]

**Ansatz for spatial decomposition:**

\[
\vec{E}(\vec{r}) = \left( E_x e^{-ik_T \vec{r}}, E_y e^{-ik_T \vec{r}}, E_z e^{-ik_T \vec{r}} \right) = \left( E_x e^{-ik_T \vec{r} \cdot \vec{e}_T}, E_x e^{-ik_Z \vec{r} \cdot \vec{e}_Z}, \ldots, \ldots \right) = \left( E_x e^{-ik_T \vec{r} \cdot \vec{e}_T}, E_y e^{-ik_Z \vec{r} \cdot \vec{e}_Z}, \ldots, \ldots \right)
\]

remark: \( E_{x,y,z} \) are complex constant

**using** \( \vec{k} \vec{r} = k_x x + k_y y + k_Z z = \left( \vec{k}_T + \vec{k}_Z \right)(\vec{r}_T + \vec{r}_Z) = \vec{k}_T \vec{r}_T + \vec{k}_Z \vec{r}_Z \) \( \text{with orthogonality relations between } z \text{ and } T \)

By inspection we find for the \( \Delta \)-operator in the Helmholtz-equation:

\[
\frac{\partial}{\partial z} \quad \rightarrow \quad -ik_z ; \quad \frac{\partial}{\partial x} \quad \rightarrow \quad -ik_x ; \quad \frac{\partial}{\partial y} \quad \rightarrow \quad -ik_x ;
\]

\[
\frac{\partial^2}{\partial z^2} \quad \rightarrow \quad -k_Z^2 ; \quad \frac{\partial^2}{\partial x^2} \quad \rightarrow \quad -k_x^2 ; \quad \frac{\partial^2}{\partial y^2} \quad \rightarrow \quad -k_y^2 \quad \text{with} \quad k^2 = |\vec{k}|^2
\]
For the all vector components $E_x, E_y, E_z$ we get for the Helmholz-equation (example $x$-comp):

\[
(\Delta + k^2)E_x (\vec{r}) = (\Delta_z + k^2_z + k^2_T)E_x (\vec{r}) = 0
\]

using \( \Delta E_x (\vec{r}) = (\Delta_T + \Delta_z)E_x (\vec{r}) = -E_x (\vec{r}) \cdot (k^2_x + k^2_y + k^2_z) \)

proves that a solutions of the form

\[
\vec{E}e^{-ik\vec{r}} = \vec{E}e^{-ik_T T} e^{-ik_z z} \quad \text{and} \quad \vec{E}e^{+ik\vec{r}}
\]

satisfies both the general Helmholz-equation and the decomposition into the

\textbf{longitudinal (z) and transvers (T) Helmholz - Equations for the E (H) - field :}

\[
(\Delta_T + k^2_T)\vec{E} (\vec{r}) = 0 \quad \text{with} \quad |\vec{k}|^2 = \omega^2 \mu \epsilon (\vec{k}_T) = k^2_T + k^2_z
\]

The Eigenfunctions \( \vec{E}(\vec{r},\vec{k}) \), resp. \( \vec{H}(\vec{r},\vec{k}) \) are called the modes of the field.

In addition the solutions must fulfill the \text{\textbf{continuity boundary conditions at the transverse interfaces}}

of the waveguide structure: (see F&KI)

Normal components : \( \vec{B}_\perp, \vec{D}_\perp \) are continuous

\[
\vec{e}_n \left( \vec{B}_2 - \vec{B}_1 \right) = 0 \quad ; \quad \vec{e}_n \left( \vec{D}_2 - \vec{D}_1 \right) \sigma_F = 0
\]

Tangential components : \( \vec{E}_\perp, \vec{H}_\perp \) are continuous

\[
\vec{e}_n \times \left( \vec{E}_2 - \vec{E}_1 \right) = 0 \quad ; \quad \vec{e}_n \times \left( \vec{H}_2 - \vec{H}_1 \right) \vec{j}_F = 0
\]
The resulting total time- and spatial dependent solutions for the E (H)-field of cylindrical, transverse homogeneous WG have the form of:

Right propagating wave:

$$\mathbf{E}e^{i(\omega t - \mathbf{k}(\omega) \mathbf{r})} = \mathbf{E}e^{i_k T \mathbf{r}} e^{i(\omega t - k_z(\omega) z)}$$

Left propagating wave:

$$\mathbf{E}e^{i(\omega t + \mathbf{k}(\omega) \mathbf{r})} = \mathbf{E}e^{i_k T \mathbf{r}} e^{i(\omega t + k_z(\omega) z)}$$

with the phase velocity

$$v_{ph,z}(\omega) = \frac{\omega}{k_z(\omega)}$$

standing wave like

z-propagating wave

The eigenvalue $k_T$ and $k_Z$ are not independent but coupled to $k(\omega)$

**reminder:**

These plane wave solutions represent not all possible wave solutions in this WG structure, because we have assumed guided waves along the z-axis in the “solution-Ansatz” and thus reduced the solution space of the problem artificially.

**Interpretation of the solutions:**

- the transverse Helmholtz-Equation defines an Eigenvalue-problem for the propagation vector $k$ for a predefined $\omega \rightarrow k(\omega)$
- $k_z$ describes the spatial dependence in the z-, $k_T$ the transverse direction
- the longitudinal propagation constant $k_z$ is influenced by the transversal solutions $k_T$, resp. by the transverse dielectric WG structure, because $k^2 = k_T^2 + k_z^2$ must hold. $k(\omega)$ is a material property.
- $k_z(\omega)$, resp. $k_T(\omega)$ defines the propagation properties as a function of wave frequency $\rightarrow$ dispersion
3.3.2 Separation of Longitudinal and Transversal Field components:

The solution of the EM-vector field has 6 vector components \((E_x, E_y, E_z, H_x, H_y, H_z)\), which are not all independent.

\[
\begin{align*}
\left(\Delta_T + k_T^2\right)\vec{E}_i(\vec{r}) &= 0 ; \\
\left(\Delta_T + k_T^2\right)\vec{H}_i(\vec{r}) &= 0 \\
\left(\Delta_z + k_z^2\right)\vec{E}_i(\vec{r}) &= 0 ; \\
\left(\Delta_z + k_z^2\right)\vec{H}_i(\vec{r}) &= 0
\end{align*}
\]

**Helmholtz-equations**

**Question:** Is it possible to solve just for a fraction of components (e.g. 2 out of 6) and derive the rest (e.g. 4) by mutual interdependencies, resp. relations?

What is the minimum number of independent and dependent field components?

Without prove we state that any vector field \(\vec{V}\) satisfies the following 2 elementary and universal relations:

\[
\begin{align*}
\vec{V} &= \vec{V}_T + \vec{V}_Z = \vec{V}_T + \vec{V}_Z,
1) \quad \vec{V}_T = \vec{e}_z \times \vec{V} \times \vec{e}_z \\
2) \quad \vec{V}_Z = \vec{e}_z \cdot \vec{V}_T \cdot \vec{e}_z
\end{align*}
\]

The equations 1) and 2) define relations between longitudinal and transversal field components \(\vec{V} \leftarrow \vec{V}_T, \vec{V}_Z\) in order to reduce the number of independent field components.

**T- and z-decomposition of vector-operations:**

Expressing the vector-operations needed by Maxwells-equations for a field that can be decomposed transversal and longitudinal components \(\vec{V} = \vec{V}_T + \vec{V}_Z = \vec{V}_T + \vec{V}_Z\):

\[
\begin{align*}
\nabla &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \nabla_T + \nabla_Z = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) + \left(\frac{\partial}{\partial z}\right)
\end{align*}
\]

**div:** \(\nabla \cdot \vec{V} = \nabla_T \cdot \vec{V}_T + \frac{\partial}{\partial z} \vec{V}_Z\)

**rot:** \(\nabla \times \vec{V} = \left(\nabla_T \cdot (\vec{V}_T \times \vec{e}_z)\right) \cdot \vec{e}_z + \left(\nabla_T \vec{V}_Z - \frac{\partial}{\partial z} \vec{V}_T\right) \times \vec{e}_z\)

(without proof, appendix B)
1) replacing the vector operators $\nabla = \nabla_T + \nabla_z$ in Maxwell's-equations, eg. for $\vec{E}$ leads to:

$$\nabla \times \vec{E} = \left( \nabla_T \cdot (\vec{E}_T \times \vec{e}_z) \right) \cdot \vec{e}_z + \left( \nabla_T E_z - \frac{\partial}{\partial z} \vec{E}_T \right) \times \vec{e}_z = -i \omega \mu \vec{H}$$

2) separating into transversal and longitudinal components:

$$\left( \nabla_T E_z - \frac{\partial}{\partial z} \vec{E}_T \right) \times \vec{e}_z = -i \omega \mu \vec{H}_T \quad (*)$$

$$\nabla_T \cdot (\vec{E}_T \times \vec{e}_z) = -i \omega \mu H_z \quad (**)$$

3) vector multiplying $\vec{e}_z \times (*)$ and using $\vec{e}_z \times \nu_T \times \vec{e}_z = \nu_T$ and $\partial l / \partial z \to -i k_z$ gives:

$$k_z \vec{E}_T - i \nabla_T E_z = -\omega \mu \left( \vec{e}_z \times \vec{H}_T \right)$$

4) from the second equation (**) by using $\nu_T \times \vec{e}_z = -\vec{e}_z \times \nu_T$, we obtain:

$$\nabla_T \cdot \left( \vec{e}_z \times \vec{E}_T \right) = i \omega \mu H_z$$

Applying the same transforms to the $\vec{H}$-field the results in the

**Maxwells-equation separated by transversal and longitudinal vector-operators**:

$$k_z \vec{H}_T - i \nabla_T H_z = \omega \varepsilon \left( \vec{e}_z \times \vec{E}_T \right)$$

$$\nabla_T \cdot \left( \vec{e}_z \times \vec{H}_T \right) = -i \omega \varepsilon E_z$$

**Interpretation:**
- relations between transversal and longitudinal field components
- 4 relations and 6 variable $\Rightarrow$ **only 2 independent field variables**
- the 4 dependent field variables can be derived from the 2 independent ones
Solution-Procedure for Maxwell’s equations:

Solve Helmholtz-Eigenvalue equations (if possible) for the longitudinal $E_z$- and $H_z$-components and determine the transversal components $E_T$ and $H_T$ by the relations.

1) Elimination of the 4 transversal field components ($E_T$, $H_T$) $\Rightarrow$ $E_z$ and $H_z$ are independent field variables (no proof)

$$\begin{align*}
\left(\Delta_z + k_T^2\right)E_z &= 0 \\
\left(\Delta_z + k_T^2\right)H_z &= 0
\end{align*}$$

2-dimensional Helmholz-Equation for $E_Z$, $H_Z$ (eigenvalue equation)

2) Solve for $E_Z$, $H_Z$ and $k_T(w)$

3) Express transversal field components $E_T$, $H_T$ by the longitudinal $E_z$, $H_z$:

$$\begin{align*}
\vec{E}_T &= \frac{1}{ik_T^2} \left\{ k_z \cdot \nabla_T E_z - \omega \mu \left( \vec{e}_z \times \nabla_T \right) H_z \right\} \\
\vec{H}_T &= \frac{1}{ik_T^2} \left\{ k_z \cdot \nabla_T H_z + \omega \varepsilon \left( \vec{e}_z \times \nabla_T \right) E_z \right\}
\end{align*}$$

longitudinal z ($k_z$, $E_z$, $H_z$) $\rightarrow$ transversal T Transform ($k_T$, $E_T$, $H_T$)

4) Using the relation between $k$, $k_T$, $k_z$ provides the calculation of the transverse propagation constant $k_z$:

$$k_T^2 = k^2 - k_z^2 = \omega^2 \varepsilon \mu (\vec{v}_T) - k_z^2$$

(3.30)

5) the boundary conditions at interfaces will provide the absolute values for $H_z$ and $E_z$
Categories of Wave Solutions (Modes):

- **TM-Wave (transverse magnetic wave) or E-wave** $E_z \neq 0, H_z \equiv 0$:
  Guided wave with only longitudinal E-field and a purely transverse magnetic field. For the solution we need only to solve the Helmholtz-equation for the $E_z$-component.

- **TE-Wave (transverse electric wave) or H-wave** $H_z \neq 0, E_z \equiv 0$:
  Guided wave with only longitudinal H-field and a purely transverse electric field. For the solution we need only solve the Helmholtz-equation for the $H_z$-component.

- **Hybrid EH- or HE-Wave (transvers electric wave) or H-wave** $E_z \neq 0, H_z \neq 0$:
  Guided wave with both longitudinal E- and H-fields (EH: $E_z$ is dominating, HE: $H_z$ is dominating). For the solution we need solve both Helmholtz-equation for the $E_z$- and $H_z$-components.

- **TEM-Wave (transverse electromagnetic Wave)** $E_z \equiv 0, H_z \equiv 0$:
  Guided wave with only transversal E- and H-fields. We can not solve the Helmholtz-equation for the $E_z$- and $H_z$-components. For TEM-Wellen we must directly solve the Helmholtz-equation for the transvers components.

  $$\left(\Delta_T + k_T^2\right)\vec{E}_T = 0$$

  TEM-wave often occur in weakly guiding WGs with small index difference between core and cladding.

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**Summary: Solutions for cylindrical waveguides**

- For waveguide the Helmholtz-equations define the eigenvalue-problem with eigenvalue $k_T$ resp. $k_z$, being the longitudinal propagation constants and the Eigenfunction of the transversal field distribution $E(r_T), H(r_T)$.

- The longitudinal Helmholtz-equations for the longitudinal components $E_z$ and/or $H_z$ are formulated for the reduction of field variables. Using the field boundary conditions the solutions are evaluated.
• Depending on the selection of the field components – only $H_z$, only $E_z$ or $E_z$ and $H_z$ combined – the corresponding mode-type is determined ($TE$-, $TM$-, or hybrid $HE$- bzw. $EH$-modes).

• The transversal components $E_T$ and $H_T$ are calculated from the primary solutions of $E_Z$ and $H_Z$.

• Enforcing the boundary conditions for the longitudinal $z$- and for the corresponding transversal $T$-components provides the necessary Eigenvalue-equation for the propagation constants $k_T$ and $k_z$.

• $TEM$-Waves are solutions of the simple transversal potential problem. This type of wave modes occurs in dielectric waveguides with very small index differences between core and cladding or in metallic multi-conductor waveguides.

• Hybrid modes are the most general solution for transverse inhomogeneous dielectric waveguides.

• Each transverse inhomogeneous, dielectric waveguide has transverse Eigensolutions, $TE$- or $TM$-Modes (no proof).
Overview of different types of technical dielectric waveguides:

For fabrication technical reasons dielectric optical waveguides are often realized by:

a) **planar deposition** (evaporation, spinning, sputtering, epitaxial growth) of planar **dielectric films on a substrate**:

![Planar deposition diagram](image)

Lateral structuring by etching, local diffusion etc.

b) **Extrusion (collapsing) of a dielectric fiber** from a heated layered cylindric perform:

Complex quasi-cylindric fiber cross-sections are possible.
3.4 **Planar waveguides**

For optoelectronic devices fabricated by planar processes the generic *dielectric planar slab waveguide* consisting of a 2-D core slab of a high refractive index $n_1$ and thickness $2d$ covered in the x-direction by two infinitely thick cladding layers of refractive index $n_2$, $n_3$ is the natural choice for integration.

Wave guiding occurs only in the x-y-plane, the wave is confined in the x-direction. We choose the z-direction as the propagation direction. The problem is homogeneous in the y-direction: $\partial / \partial y = 0$

\[ E_z \equiv 0, \quad H_z \neq 0 \]

1) Solve the Helmholz-Equation for $H_z$-component in the loss-less medium:

\[ k^2 - n_i^2 \beta^2 = 0 \]

with the longitudinal propagation constant $k_z = 0$

\[ n_k(\omega) = \frac{\omega}{c} = \frac{\lambda}{\lambda_{\text{vacuum}}} \]

\[ k^2 = n_1^2, \quad n_2^2, \quad n_3^2 \]

\[ E_Z, H_Z \]

\[ \beta \quad \text{and} \quad n_i (\forall i = 1, 2) \]
3.5 Ridge (Rib) Waveguides

Practical planar optical waveguides (confinement in x-direction) need an additional lateral (y) confinement to separate the optical channels from each other in the film plane.

- 2-dimensional dielectric confinement in the 2 transverse directions x and y

![Diagram of 2-dimensional dielectric confinement](image)

- Technical realizations of 2-dimensional film waveguides (WG):
  - (a) strip WG (Streifenleiter),
  - (b) embedded strip WG (eingebetteter Streifenleiter),
  - (c) rib- or ridge WG (Rippenwellenleiter),
  - (d) loaded strip WG (aufliegender Streifenleiter)

Legend: the darker the grey-scale, the larger the refractive index.
3.6 Optical Glass Fibers

Optical glass fibers are the most important waveguides for long transmission distances in optical communication and therefore attenuation and dispersion effects are critical.

Fiber fabrication is based on a preform drawing process that lends itself to **cylindrical wave guides** with a high index core cylinder $n_1$ (SMF: $a\sim4\mu m$, MMF: $a\sim25-31\mu m$) surrounded by a low index $n_2$ cylindrical cladding layer of about $250\mu m$ diameter.

The **step index fiber** with an abrupt index difference $\Delta n=(n_2-n_1)$ is the simplest transverse index profile $n(r,\phi)$.

Obviously a cylindrical coordinate system $(z, r, \phi)$ is the appropriate representation with $z$ as the propagation direction and $r$, $\phi$ as the transverse coordinates.

![Step index glass fibers diagram](image)

Instead of using the cartesihan coordinates $(x, y)$ in the transverse direction we will use the cylindrical system $(r, \phi)$. For the formulation of Helmholtz-equations we have to transform the vector operators in to the cylindrical coordinates:
Coordinate transformation \( x, y, z \Rightarrow r, \phi, z: \)

The transform is straightforward but lengthy, so only the starting point is given.

\[
x = r \cdot \cos \phi \quad ; \quad y = r \cdot \sin \phi \quad ; \quad z = z
\]

\[
r = \sqrt{x^2 + y^2} \quad ; \quad \phi = \arctan(y/x) \quad ; \quad z = z
\]

\(
\Delta - \text{Operator} : \quad f(x,y)
\)

\[
\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial r} \left( \frac{\partial r}{\partial x} \right)^2 + \frac{\partial f}{\partial r} \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 f}{\partial \phi^2} \left( \frac{\partial \phi}{\partial x} \right)^2 + \frac{\partial f}{\partial \phi} \frac{\partial^2 \phi}{\partial x^2} \quad \text{analog für} \quad \frac{\partial^2 f}{\partial y^2}
\]

\[
\rightarrow \quad \Delta_T = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2}{\partial \phi^2}
\]

### 3.6.1 Vector field solutions for the step-index fiber

The step-index fiber has the only simple index profile where the field can be calculated analytically in terms of cylindric Bessel-functions.

1) Transformation of Helmholtz-equations for the longitudinal components into cylindrical coordinates:

\[
\begin{align*}
\left( \Delta + k_T^2 \right) E_z(x,y) &= 0 \\
\left( \Delta + k_T^2 \right) H_z(x,y) &= 0 
\end{align*}
\]

\[
\Delta_T = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2}{\partial \phi^2}
\]

with the definition for the transverse wave number \( k_T \):

\[
k_T^2 = k^2 - \beta^2 = \omega^2 \mu \varepsilon(r_T) - \beta^2
\]
2) Solution by coordinate separation:

\[ E_z(r, \phi) = \begin{cases} A & R(r) \cdot \phi(\phi) \\ B & \end{cases} \]

Solution-"Ansatz" with radial and azimuthal separation

Insertion of the "Ansatz" and separating into \( R(r) \) and \( \phi(\phi) \) leads to 2 second order, uncoupled differential equations:

\[
\frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} + \frac{R}{r^2} \frac{\partial^2 \phi}{\partial \phi^2} + k_0^2 R \cdot \phi = 0 \quad \text{multiply on both sides} \quad \rightarrow \quad \frac{r^2}{R} \frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} + k_0^2 R \cdot \phi = 0 \quad \text{multiply on both sides} \quad \rightarrow \quad \frac{m^2 = \text{constant}}{R(r) = 0} \quad m=\text{constant}
\]

\[
\frac{r^2}{R} \frac{\partial^2 R}{\partial r^2} + \frac{r}{R} \frac{\partial R}{\partial r} + r^2 k_0^2 = -\frac{1}{\phi} \frac{\partial^2 \phi}{\partial \phi^2} = m^2 = \text{constant}
\]

2 decoupled differential equations for \( R(r) \) and \( \phi(\phi) \)

\( m \) is an undefined constant.

3) Harmonic azimuthal solutions for \( \phi(\phi) \):

\[
\phi(\phi) = \begin{cases} \sin(m \cdot \phi) \\ \cos(m \cdot \phi) \end{cases} \quad \text{for symmetry reason: } m=0, 1, 2, 3 \quad \text{integer are possible}
\]

The radial solutions must be radial periodic and symmetric with respect to the z-axis and \( 2m \) indicates the number of radial nodes of the field.
4) Bessel-functions for radial solutions for $R(r)$:

The radial Bessel-solutions depend on $k_T$ and $m$

<table>
<thead>
<tr>
<th>function</th>
<th>physical interpretation</th>
<th>Cartesian correspondence</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_T : \text{real}$</td>
<td>standing cylindrical wave</td>
<td>$\cos(k_x x)$</td>
</tr>
<tr>
<td>$R(r) = J_m(k_T r)$</td>
<td>Zylinderfunktion 1. Art $\rightarrow J_m : \text{Besselfunktion}$</td>
<td></td>
</tr>
<tr>
<td>$R(r) = N_m(k_T r)$</td>
<td>Zylinderfunktion 2. Art $\rightarrow N_m : \text{Neumannfunktion}$</td>
<td>$\sin(k_x x)$</td>
</tr>
<tr>
<td>$R(r) = H^{(1)}_m(k_T r) = J_m(k_T r) + i N_m(k_T r)$</td>
<td>propagating cylindrical wave</td>
<td>$e^{+ik_x x}$</td>
</tr>
<tr>
<td>$R(r) = H^{(2)}_m(k_T r) = J_m(k_T r) - i N_m(k_T r)$</td>
<td>Zylinderfunktion 3. Art $\rightarrow H^{(1,2)}_m : \text{Hankelfunktionen}$</td>
<td>$e^{-ik_x x}$</td>
</tr>
<tr>
<td>$k_T \rightarrow -i k_T' : \text{imaginary}$</td>
<td>growing cylindrical wave</td>
<td>$e^{k_x x}$</td>
</tr>
<tr>
<td>$R(r) = I_m(k_T r) = i^m J_m(-i k_T' r)$</td>
<td>$\rightarrow I_m : \text{modifizierte Besselfunktionen}$</td>
<td></td>
</tr>
<tr>
<td>$R(r) = K_m(k_T r) = \frac{\pi}{2} (-i)^{m+1} H^{(2)}_m(-i k_T' r)$</td>
<td>decaying cylindrical wave</td>
<td>$e^{-k_x x}$</td>
</tr>
</tbody>
</table>

The type of solutions of the Bessel-differential equation and the cartesian correspomencies for the 3 layer film waveguide

For cylindrical functions see at summary at the end of the chapter.
We consider the transverse wave number \( k_{Ti}^2 = k_i^2 - \beta^2 \) corresponding to medium \( i \).

For mode confinement as before the eigenvalue \( \beta \) is restricted to the interval \( k_2 = k_0' n_2 < \beta < k_1 = k_0' n_1 \).

General solution for the longitudinal components \( E_z \) and \( H_z \) for a homogeneous medium section:

\[
\begin{align*}
E_z &= \left\{ A_0 \right\} Z_0(k_z r) + \sum_{m=1}^{\infty} \left\{ A_m \right\} Z_m(k_z r) \cdot \cos(m\phi) + \left\{ B_m \right\} Z_m(k_z r) \cdot \sin(m\phi) \\
H_z &= \left\{ B_0 \right\} Z_0(k_z r) + \sum_{m=1}^{\infty} \left\{ B_m \right\} Z_m(k_z r) \cdot \cos(m\phi) + \left\{ A_n \right\} Z_n(k_z r) \cdot \sin(m\phi)
\end{align*}
\]

\( Z_m(...) \) is a cylinder function from the table for \( R(r) \) depending on medium \( i \) with \( n_i \) (core or cladding) and argument \( m \).

5) Continuity conditions of the transversal field at \( r = a \) for all \( \phi = 0 \ldots 2\pi \):

The formulation of the continuity requires the additional calculation of the transverse field components \( E_r, E_\phi, H_r \) und \( H_\phi \) in cylinder coordinates (without proof):

\[
\begin{align*}
E_r &= \frac{1}{ik_z^2} \left\{ \beta \cdot \frac{\partial}{\partial r} E_z + \omega \mu \cdot \frac{1}{r} \frac{\partial}{\partial \phi} H_z \right\} \\
E_\phi &= \frac{1}{ik_z^2} \left\{ \beta \cdot \frac{1}{r} \frac{\partial}{\partial \phi} E_z - \omega \mu \cdot \frac{1}{r} \frac{\partial}{\partial r} H_z \right\} \\
H_r &= \frac{1}{ik_z^2} \left\{ \beta \cdot \frac{\partial}{\partial r} H_z - \omega \varepsilon \cdot \frac{1}{r} \frac{\partial}{\partial \phi} E_z \right\} \\
H_\phi &= \frac{1}{ik_z^2} \left\{ \beta \cdot \frac{1}{r} \frac{\partial}{\partial \phi} H_z + \omega \varepsilon \cdot \frac{1}{r} \frac{\partial}{\partial r} E_z \right\}
\end{align*}
\]

\( E_{z1} - E_{z2} = 0 \quad H_{z1} - H_{z2} = 0 \)

\( E_{\phi1} - E_{\phi2} = 0 \quad H_{\phi1} - H_{\phi2} = 0 \)

**Boundary conditions**
The boundary conditions lead to an infinite set of equations for \( A_0, B_0, A_1^m, A_2^m, B_1^m \) and \( B_2^m \), \( m=1,2,3,\ldots\infty \), but symmetry and rotation invariance properties of the solutions reduce the solution space to:

- **core \( \mathbb{Q} \):** \( n = n_1, \quad k_1 > \beta, \quad r < a \)

\[
E_z(r, \varphi) = A_j \cdot J_m(k_1 r) \cdot \cos(m \varphi + \varphi_0) \\
H_z(r, \varphi) = B_j \cdot J_m(k_1 r) \cdot \sin(m \varphi + \varphi_0)
\]
oscillatory transverse wave solution;

- **cladding \( \mathbb{Q} \):** \( n = n_2, \quad k_2 < \beta, \quad r > a \)

\[
E_z(r, \varphi) = A_2 \cdot K_m(k_2 r) \cdot \cos(m \varphi + \varphi_0) \\
H_z(r, \varphi) = B_2 \cdot K_m(k_2 r) \cdot \sin(m \varphi + \varphi_0)
\]
decaying, transverse confined, evanescent wave solution

\[ \text{Reduction to 4 terms: } A_1, A_2, B_1, B_2 \text{ are the desired solutions for a particular } m \text{ and a particular wave excitation as boundary condition.} \]

The solutions are inserted into the tangential boundary conditions and define a set of equations for \( A_1, A_2, B_1, B_2 \) and the unknown propagation constant \( \beta(\omega) = k_2(\omega) \) as eigenvalue:

Using similar substitutions as in the case of the slab waveguide

\[
\xi = a \cdot k_1 = a \cdot \sqrt{k_1^2 - \beta^2} \quad \eta = a \cdot k_2 = a \cdot \sqrt{\beta^2 - k_2^2}
\]
we obtain (without proof) following system of eq.

\[
\begin{bmatrix}
J_m(\xi) & 0 & -K_m(\eta) & 0 \\
0 & J_m(\eta) & 0 & -K_m(\xi) \\
\pm \frac{\beta}{\xi} J_m(\xi) & \frac{\beta}{\xi} J'_m(\xi) & \pm \frac{\beta}{\xi} K_m(\eta) & \frac{\beta}{\xi} K'_m(\eta) \\
\frac{\beta}{\xi} J'_m(\xi) & \frac{\beta}{\xi} J_m(\xi) & \frac{\beta}{\xi} K_m(\eta) & \frac{\beta}{\xi} K'_m(\eta)
\end{bmatrix}
\begin{bmatrix}
A_1 \\
B_1 \\
A_2 \\
B_2
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]
for \( m=1,2,3,\ldots\infty \) and \( Z'(x) = \frac{\partial Z}{\partial x} \)
The sign ± corresponds to the cos- \((\varphi_0 = 0)\) resp. sin-solution \((\varphi_0 = \pm \pi / 2)\) for \(E_z\).

Nontrivial solutions for \(A_1, A_2, B_1, B_2\) only exist if the determinant of the homogeneous system vanishes, leading to the eigenvalue equation for \(\xi, \eta:\)

\[
\begin{vmatrix}
 J_m(\xi) & 0 & -K_m(\eta) & 0 \\
 0 & J_m(\xi) & 0 & -K_m(\eta) \\
 \frac{\pm \beta_m}{\xi} J_m(\xi) & \frac{\pm \beta_m}{\xi} K_m(\eta) & \frac{\pm \beta_m}{\xi} K_m(\eta) \\
 \frac{\pm \beta_m}{\xi} J_m(\xi) & \frac{\pm \beta_m}{\xi} K_m(\eta) & \frac{\pm \beta_m}{\xi} K_m(\eta)
\end{vmatrix} = 0
\]

\(\Rightarrow\) Eigenvalue equation \(f(\eta, \xi, m)=0\)

**Characteristic equation, resp. Eigenvalue equation:**

\[
\left\{ k_1^2 \cdot \tilde{J}_m(\xi) + k_2^2 \cdot \tilde{K}_m(\eta) \right\} \cdot \left\{ \tilde{J}_m(\xi) + \tilde{K}_m(\eta) \right\} - m^2 \beta^2 \left( \frac{1}{\xi^2} + \frac{1}{\eta^2} \right)^2 = 0
\]

with the definitions: \(\tilde{J}_m(\xi) = \frac{J_m(\xi)}{\xi \cdot J_m(\xi)}\); \(\tilde{K}_m(\eta) = \frac{K_m(\eta)}{\eta \cdot K_m(\eta)}\)

In analogy to the sym. Planar WG we make use of a single structure parameter or **fiber parameter** \(V(\omega)\) to eliminate either \(\xi\) or \(\eta\):

\[V(\omega) = a \cdot \sqrt{k_1^2 - k_2^2} = a \cdot k_0 \cdot \sqrt{n_1^2 - n_2^2} = a \cdot k_0 \cdot NA = \sqrt{\xi^2 + \eta^2} ; \quad k_0 = \frac{2\pi}{\lambda_o} = \frac{\omega}{c_o}\]

**6) Formal solution procedure for \(\beta(\omega)\):**

- chose an integer \(=0, 1, 2, \ldots\) and \(\omega\)
- eliminate \(\eta\) with the above eigenvalue eq. by using \(V(\omega)\) and find the zero \(\xi_p\). There a \(p\) zeros of the eigenvalue equation for a given \(m\)
- from \(\xi_p\) we determine \(\beta_{mp}(\omega)\) of the modes characterized by the numbers \(m,p\)
- for \(m\) and \(p\) we associate a particular modal solution of **mode** \(X_{mp}\). with \(X= HE-, EH-, TE-\) or **TM-modes**.
Classification of Modes $X_{pm}$:

1. class: $m=0$ (azimuthally homogeneous), and $\beta=0$ (no cut-off)

1) for $m=0$ the 4x4 determinant splits into two independent 2x2 sub-determinants for $A_1$, $A_2$ (TE) and $B_1$, $B_2$ (TM).

$$
\text{TE- resp. TM-modes}
$$

$$
\begin{align*}
&\begin{bmatrix}
J_m(\xi) & 0 & -K_m(\eta) & 0 \\
0 & J_m(\xi) & 0 & -K_m(\eta) \\
0 & \frac{\partial J_m}{\partial \xi}(\xi) & 0 & \frac{\partial K_m}{\partial \eta}(\eta) \\
\frac{\partial J_m}{\partial \xi}(\xi) & 0 & \frac{\partial K_m}{\partial \eta}(\eta) & 0
\end{bmatrix}
\begin{bmatrix}
A_1 \\
B_1 \\
A_2 \\
B_2
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix} \\
\begin{bmatrix}
J_m(\xi) & -K_m(\eta) \\
\frac{\partial J_m}{\partial \xi}(\xi) & \frac{\partial K_m}{\partial \eta}(\eta)
\end{bmatrix}
\begin{bmatrix}
A_1 \\
A_2
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\end{align*}
$$

2) $m=0$ TE- and TM-modes have radial symmetry

3) $\beta=0$ represents a mode which has its cutoff at $\omega=0$ $\Rightarrow$ HE11-mode

**Dispersion relation $\beta(\omega)$ for $TE_{op}$-modes** $(E_z=0, E_r=0, H_\phi=0,)$:

$$
\begin{align*}
\{J_m(\xi)+K_m(\eta)\} = 0 \text{ using } J_0'(\xi) = -J_1(\xi) \text{ and } K_0'(\eta) = -K_1(\eta) \Rightarrow \\
\frac{J_1(\xi)}{\xi} + \frac{K_1(\eta)}{\eta} = 0 \Rightarrow \beta_{TE_{op}}(\omega)
\end{align*}
$$

**Dispersion relation $\beta(\omega)$ for $TM_{op}$-modes** $(H_z=0, H_r=0, E_\phi=0,)$:

$$
\begin{align*}
\{k_1^2 J_m(\xi) + k_2^2 K_m(\eta)\} = 0 \Rightarrow \\
\frac{k_1^2 J_1(\xi)}{\xi} + \frac{k_2^2 K_1(\eta)}{\eta} = 0 \Rightarrow \beta_{TM_{op}}(\omega)
\end{align*}
$$
2. class: m ≠ 0, and β ≠ 0

general case \( \Rightarrow \) hybrid modes \((E_z \neq 0, H_z \neq 0)\)

classification of modes by:

a) inspection

\[
\lim_{v \to \infty} \left\{ \frac{E_z}{H_z} \right\} = \begin{cases} 0 & EH \\ \infty & HE \end{cases}
\]

TE-like because \( H_z \) is dominant

TM-like because \( E_z \) is dominant

b) approximation of weak guiding \( n_1 \approx n_2 \approx n_{\text{eff}} \)

The general eigenvalue equation simplifies to

\[
\left\{ \tilde{J}_m(\xi) + \tilde{K}_m(\eta) \right\} \mp m \cdot \left( \frac{1}{\xi^2} + \frac{1}{\eta^2} \right) = 0
\]

sign convention: + for \( EH_{mp}^- \) and - for \( HE_{mp}^- \) modes

and further to

\[
\frac{J_{m-1}(\xi)}{\xi \cdot J_m(\xi)} - \frac{K_{m-1}(\eta)}{\eta \cdot K_m(\eta)} \approx 0 \quad \Rightarrow \quad HE_{mp}^- \\
\frac{J_{m+1}(\xi)}{\xi \cdot J_m(\xi)} + \frac{K_{m+1}(\eta)}{\eta \cdot K_m(\eta)} \approx 0 \quad \Rightarrow \quad EH_{mp}^-
\]

\( \Rightarrow \) Approximate dispersion relation \( \beta(\omega) \) for hybrid modes
Field cross-sections of guided modes in step-index glass fibers:

The most relevant mode is the $\text{HE}_{11}$ ($m=1$, $p=1$) with a zero-frequency cut-off.

Dispersion curves $n_{\text{eff}}(V)$ of guided modes in step-index glass fibers:
Dispersion in step-index Glassfibers:

A nonlinear dispersion $\beta(\omega)$ leads to frequency dependent group velocities $v_{gr}(\omega)$ and dispersion $D(\omega)$

Cutoff-condition $V_{mp}$ of mode $(m,p)$:
- for $V < V_{pm} = \xi @ \eta=0$ no pm-mode can exist
- for $V > V_{pm}$ the pm-mode exist and is described by the dispersion relation $\beta_{pm}(\omega)$

At cutoff, the modes do not decay anymore in the cladding

Observe that modes tend to build groups of similar dispersion curves

Cutoff-condition $\eta = a\sqrt{\beta^2 - k_2^2} \to 0$ for different modes: (without proof)

$m=0$: TE$_{0p}$, TM$_{0p}$

$J_0(\xi) = 0$

$m=1$: HE$_{1p}$, EH$_{1p}$

$J_1(\xi) = 0$

$m>1$: EH$_{mp}$

$J_m(\xi) = 0$

$m>1$: HE$_{mp}$

$$\left(\frac{n_1^2}{n_2^2}+1\right) \cdot J_{m-1}(\xi) = \frac{\xi}{m-1} \cdot J_m(\xi)$$
Conclusions:

1) the **fundamental mode** is the **HE$_{11}$-mode**, the fiber is fundamental (single) mode for $0<V(\omega)<2.405$ (no intermodal dispersion occurs)

2) the fundamental mode exists even at $\omega=0$, no cut-off

3) **TE- and TM-modes** are not degenerate (due to rotational symmetry), however they are degenerate at cut-off

4) **Hybrid modes** are 2-times degenerate, because there exist 2 radial solutions

\[ \cos(n\varphi) \text{ and } \cos\left(n\left[\varphi - \frac{\pi}{2n}\right]\right) \] (90°-rotation. eg. orthogonal polarizations)

**Approximation of Number of modes vers. V:**

Question: how many modes $N$ exist for a certain $V$, resp. $\omega$ ?

\[ N \approx \frac{V^2}{2} = \frac{a^2 k_0^2 \Delta n}{2} \] (without proof)
3.6.2 Scalar Approximation – LP-modes (linear polarized modes) (optional)

The dispersion curves of the step-index fiber for weak guiding (quasi-TEM-modes) show that the different modes tend to form groups (degeneracy) with similar $\beta(\omega)$-curves – this tendency becomes more pronounced for weak guiding ($n_1 \sim n_2 \sim n_{\text{eff}}$), resp.

$$\frac{\lambda_0 \cdot \frac{\partial n}{\partial r}}{n} \ll 1 \quad \text{and} \quad k_1 \sim k_2 \sim \beta$$

The following mode-clustering into mode groups {...} with almost identical $\beta(\omega)$–characteristics can be observed:

- $\{\text{HE}_{11}\}$
- $\{\text{TE}_{01}, \text{TM}_{01}, \text{HE}_{21}\}$
- $\{\text{HE}_{31}, \text{EH}_{11}\}$
- $\{\text{HE}_{12}\}$
- $\{\text{HE}_{41}, \text{EH}_{21}\}$
- $\{\text{TE}_{02}, \text{TM}_{02}, \text{HE}_{22}\}$

Generating LP-modes in the weak guiding approximation:

Modes having virtually the same degenerate dispersion curve $n_{\text{eff}}(V)$ can not be distinguished. The degenerated modes can be represented by a LP-mode which is the superposition of the mode-group forming e.g.

$$\{\text{HE}_{11}\}, \{\text{TE}_{01}, \text{TM}_{01}, \text{HE}_{21}\}, \{\text{HE}_{31}, \text{EH}_{11}\}, \{\text{HE}_{12}\}, \{\text{HE}_{41}, \text{EH}_{21}\}, \{\text{TE}_{02}, \text{TM}_{02}, \text{HE}_{22}\} \text{ or } \ldots$$

In general

$$\text{HE}_{m+1,p} \approx \text{EH}_{m-1,p}, \quad \text{TE}_{0,p} \approx \text{TM}_{0,p} \approx \text{HE}_{2,p}$$

leading to LP-modes:

$$LP_{0p} \triangleq \text{HE}_{1,p}$$

$$LP_{m,p}^y \triangleq \text{EH}_{m-1,p} + \text{HE}_{m+1,p} \quad \text{x= ? \quad y= ?}$$

$$LP_{m,p}^x \triangleq \text{EH}_{m-1,p} - \text{HE}_{m+1,p}$$

and reversed:

$$\text{EH}_{m-1,p} \triangleq LP_{m,p}^y + LP_{m,p}^x \quad ; \quad \text{HE}_{m+1,p} \triangleq LP_{m,p}^y - LP_{m,p}^x$$
Examples of LP-modes:

Example of LP-modes created from hybrid modes for weakly guiding WG.

For higher order LP-modes we show only typical cases of degeneracy.

Considering all possible degeneracy for the 4 \( \text{LP}_{11} \)-modes:

Example of LP-modes created from hybrid modes for weakly guiding WG.

For higher order LP-modes we show only typical cases of degeneracy.
Dispersion relation and scalar representation of LP-mode approximations:

Scalar Calculation using LP-modes

Starting from the general eigenvalue equation

\[ \{ k_1^2 \cdot \tilde{J}_m(\xi) + k_2^2 \cdot \tilde{K}_m(\eta) \} \cdot \{ \tilde{J}_m(\xi) + \tilde{K}_m(\eta) \} - m^2 \beta^2 \left( \frac{1}{\xi^2} + \frac{1}{\eta^2} \right)^2 = 0 \]

we obtain in the weak guiding approximation \( k_1 \cong k_2 \cong \beta \) the simplified relation

\[ \{ \tilde{J}_m(\xi) + \tilde{K}_m(\eta) \}^2 - m^2 \left( \frac{1}{\xi^2} + \frac{1}{\eta^2} \right)^2 = 0 \]

\[ \{ \tilde{J}_m(\xi) + \tilde{K}_m(\eta) \} - m \cdot \left( \frac{1}{\xi^2} + \frac{1}{\eta^2} \right) = 0 \]

and we end up with:

\[
\frac{\xi \cdot J_{\ell-1}(\xi)}{J_{\ell}(\xi)} = \frac{\eta \cdot K_{\ell-1}(\eta)}{K_{\ell}(\eta)} \quad \forall \ \ell = \begin{cases} 
1 & TE, TM \\
m + 1 & EH \\
m - 1 & HE
\end{cases}
\]

Eigenvalue equation for weakly guided LP-modes

The purely transverse field configuration \((E_x, H_y)\) resp. \((E_y, H_x)\) of the \(LP_{\ell \rho}\)-modes are:

- **core \( \Theta \):** \( n = n_1, \ k_1 > \beta, \ r < a \)

\[
E_x(r, \varphi) = A_\ell \cdot J_\ell \left( \frac{\xi}{a} r \right) \cdot \cos(\ell \varphi + \varphi_0)
\]

\[
H_y(r, \varphi) = E_x(r, \varphi) \cdot \sqrt{\frac{\varepsilon_{\text{eff}}}{\mu_0}} \quad ; \quad \varepsilon_{\text{eff}} = n_1^2 \approx n_2^2 \quad (3.186)
\]
**cladding**: \( n = n_2 \), \( k_2 < \beta \), \( r > a \)

\[
E_x(r, \varphi) = A_l \cdot \frac{J_l(\xi)}{K_l(\eta)} \cdot K_l\left(\frac{2}{a}r\right) \cdot \cos(\ell\varphi + \varphi_0)
\]

\[
H_y(r, \varphi) = E_x(r, \varphi) \cdot \sqrt{\frac{\varepsilon_{\text{eff}}}{\mu_0}}
\]

For the \((E_x, H_y)\)-field we can assume \( \varphi_0 = 0 \), for the \((E_y, H_x)\)-field must use \( \varphi_0 = -\pi/2 \) and the expression for \( H_x \) becomes a negative sign for having in both cases a Pointing vector \( \mathbf{E} \times \mathbf{H} \) in the positive z-direction.

**Examples of \( LP_\ell p \)-modes:**

Mode profile \( E_\ell(r, \varphi) \) of the \( LP_{21} \)-mode for a step-index fiber with a core radius \( a \).

Typical profile of the transverse \( E \)-field \( E_x(r, \varphi) \) for a step-index fiber with a fiberparameter \( V = 5 \).
Literature:


Graphical summary of Cylinder Functions:

Hyperbolic functions:

Bessel functions (first kind):

Bessel function (second kind):

Hankel function: